Method of Frobenius

Consider the following equation

$$\mathcal{L}y = R(x)\frac{d^2y}{dx^2} + \frac{1}{x}P(x)\frac{dy}{dx} + \frac{1}{x^2}Q(x)y = 0 \tag{1}$$

where $R(x)$, $P(x)$ and $Q(x)$ are analytic at $x = 0$ and $R(x) \neq 0$ in some interval containing $x = 0$. Without losing generality, let us assume $R(0) = 1$.

Since the functions $P(x)$, $Q(x)$ and $R(x)$ are analytic at $x = 0$, then can be expanded in power series, i.e.,

$$P(x) = P_0 + P_1x + P_2x^2 + \cdots \tag{2}$$

$$Q(x) = Q_0 + Q_1x + Q_2x^2 + \cdots \tag{3}$$

$$R(x) = 1 + R_1x + R_2x^2 + \cdots \tag{4}$$

Also, since $R(0) = 1$ and therefore $x = 0$ is at least a regular singular point of equation (1), the solution can be written as

$$y = x^s \sum_{k=0}^{\infty} a_k x^k \tag{5}$$

where $a_0 \neq 0$ and s is to be determined. Substitute this solution into equation (1), one obtain

$$\begin{aligned}\mathcal{L}y \equiv\ & [s(s-1) + P_0s + Q_0]\, a_0x^{s-2} \\ & + \{[s(s+1) + P_0(s+1) + Q_0]\, a_1 \\ & \qquad + [R_1s(s-1) + P_1s + Q_1]\, a_0\}\, x^{s-1} \\ & + \{[(s+1)(s+2) + P_0(s+2) + Q_0]\, a_2 \\ & \qquad + [R_1(s+1)s + P_1(s+1) + Q_1]\, a_1 \\ & \qquad + [R_2s(s-1) + P_2s + Q_2]\, a_0\}\, x^s \\ & \qquad \cdots \end{aligned} \tag{6}$$

Let

$$f(s) = s(s-1) + P_0s + Q_0 \tag{7}$$

$$g_n(s) = R_n(s-n)(s-n-1) + P_n(s-n) + Q_n \tag{8}$$

Thus,

$$\begin{aligned}\mathcal{L}y &= f(s)a_0x^{s-2} + [f(s+1)a_1 + g_1(s+1)a_0]\,x^{s-1} \\ &\quad + [f(s+2)a_2 + g_1(s+2)a_1 + g_2(s+2)a_0]\,x^{s} \\ &\quad \cdots \\ &\quad + \left[f(s+k)a_k + \sum_{i=1}^{k} g_i(s+k)a_{k-i}\right] x^{s+k-2} \\ &\quad \cdots \\ &= 0 \end{aligned} \tag{9}$$

From the coefficient of x^{s-2}, since $a_0 \neq 0$,

$$f(s) = s^2 + (P_0 - 1)s + Q_0 = 0 \tag{10}$$

This is the **Indicial Equation**! Let us further consider the coefficient of x^{s-1}

$$f(s+1)a_1 + g_1(s+1)a_0 = 0 \tag{11}$$

If $f(s+1) \neq 0$, then the above equation can be re-written as

$$a_1 = -\frac{g_1(s+1)}{f(s+1)}a_0 \tag{12}$$

Similarly, if $f(s+2) \neq 0$, the coefficient of x^s yields

$$a_2 = -\frac{g_1(s+2)}{f(s+2)}a_1 - \frac{g_2(s+2)}{f(s+2)}a_0 \tag{13}$$

In general the following formulas can be derived from the coefficient of x^{s+k-2}:

$$a_k = -\frac{1}{f(s+k)}\sum_{i=1}^{k} g_i(s+k)a_{k-i} \tag{14}$$

where $k = 1, 2, 3, \cdots \Rightarrow$ If $f(s+k) \neq 0$, each a_k can be expressed in terms of a_0.

Therefore, if the two values of s determined by the Indicial Equation are distinct and they are not differed by an integer, then two series solutions of equation (1) in the form of equation (5) can be obtained.

Exceptional Cases

Let s_1 and s_2 be the roots of indicial equation.

1. $s_1 = s_2$

   Only one series solution can be obtained by the above method.

2. $s_1 = s_2 + \ell$ and ℓ is a positive integer.

   For a_0

$$f(s) = (s - s_1)(s - s_2) \tag{15}$$

For a_k and $k = 1, 2, \cdots$

$$f(s + k) = (s + k - s_1)(s + k - s_2) \tag{16}$$

When $s = s_1$,

$$f(s_1 + k) = k(s_1 - s_2 + k) = k(k + \ell) > 0 \tag{17}$$

for $k = 1, 2, 3, \cdots$.

When $s = s_2$,

$$f(s_2 + k) = (s_2 - s_1 + k)k = (k - \ell)k = 0 \tag{18}$$

for $k = \ell$. Thus, the 2nd solution cannot be obtained with the above method.

However, if the numerator of the RHS of equation (14) is also zero, there will still be a second solution corresponding to s_2 in the form of equation (5).

Specifically, from equation (14)

$$f(s + k)a_k = -\sum_{i=1}^{k} g_i(s + k)a_{k-i} \tag{19}$$

If both $f(s_2 + k) = 0$ and also $\sum_{i=1}^{k} g_i(s + k)a_{k-i} = 0$ when $k = \ell$ and $s = s_2$, then

$$0 \cdot a_\ell = 0 \tag{20}$$

Thus, a_ℓ is undetermined and can be any arbitrary constant. Equation (14) is still valid for $k = \ell + 1, \ell + 2, \cdots$.

[Example] Exception to exceptional case No. 2

$$\mathcal{L}y \equiv x^2 \frac{dy^2}{dx^2} + (x^2 + x)\frac{dy}{dx} - y = 0$$

Let

$$y = \sum_{k=0}^{\infty} a_k x^{k+s}$$

$$\mathcal{L}y \equiv \sum_{k=0}^{\infty} (k+s)(k+s-1)a_k x^{k+s} + \sum_{k=0}^{\infty} (k+s)a_k x^{k+s+1} \qquad (k+1=\ell)$$

$$+ \sum_{k=0}^{\infty} (k+s)a_k x^{k+s} - \sum_{k=0}^{\infty} a_k x^{k+s}$$

$$= \sum_{k=0}^{\infty} \left[(k+s)(k+s-1) + (k+s) - 1\right] a_k x^{k+s} + \sum_{\ell=1}^{\infty} (\ell+s-1)a_{\ell-1} x^{\ell+s}$$

$$= (s^2-1)a_0 x^s + \sum_{k=1}^{\infty} \left\{ a_k \left[(k+s)^2 - 1\right] + a_{k-1}(k+s-1) \right\} x^{k+s} = 0$$

Indicial Equation:

$$s^2 - 1 = 0$$

$$s_1 = 1 \qquad s_2 = -1 \qquad (s_1 - s_2 = 2)$$

Recurrence Formula

$$(k+s+1)(k+s-1)a_k = -(k+s-1)a_{k-1}$$

When $s = 1$,

$$a_k = -\frac{a_{k-1}}{k+2} \qquad k = 1, 2, 3, \cdots$$

$$a_1 = -\frac{a_0}{3}; \qquad a_2 = -\frac{a_1}{4} = \frac{a_0}{4 \cdot 3}; \qquad a_3 = -\frac{a_0}{5 \cdot 4 \cdot 3}; \qquad \cdots$$

The first solution is

$$y_1(x) = a_0 \left(x - \frac{x^2}{3} + \frac{x^3}{4 \cdot 3} - \frac{x^4}{5 \cdot 4 \cdot 3} + \cdots \right)$$

$$= \frac{2a_0}{x} \left[-1 + x + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \right) \right]$$

$$= a_0 \frac{2}{x} \left(-1 + x + e^{-x} \right)$$

When $s = -1$, the the recurrence formula is

$$k(k-2)a_k = -(k-2)a_{k-1} \qquad k = 1, 2, 3, \cdots$$

$$k = 1 \qquad \Rightarrow \qquad -1 \cdot a_1 = a_0 \qquad \Rightarrow a_1 = -a_0$$
$$k = 2 \qquad \Rightarrow \qquad 0 \cdot a_2 = 0 \cdot a_1 \qquad \Rightarrow a_2 \text{ arbitrary}$$

For $k = 3, 4, 5, \cdots$,

$$a_k = -\frac{a_{k-1}}{k}$$

$$a_3 = -\frac{a_2}{3}; \qquad a_4 = \frac{a_2}{4 \cdot 3}; \qquad a_5 = -\frac{a_2}{5 \cdot 4 \cdot 3}; \qquad \cdots$$

Thus,

$$\begin{aligned} y_2(x) &= a_0 x^{-1}(1-x) + a_2 x^{-1}\left(x^2 - \frac{x^3}{3} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4 \cdot 3} + \cdots\right) \\ &= a_0 x^{-1}(1-x) + 2a_2 x^{-1}\left[(-1+x) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots)\right] \\ &= a_0 \frac{1-x}{x} + a_2 \frac{2}{x}(e^{-x} + x - 1) \end{aligned}$$

Since the second term is the same as y_1, the complete solution can be written as

$$y = c_1 \frac{1-x}{x} + c_2 \frac{e^{-x} + x - 1}{x}$$

or

$$= \tilde{c}_1 \frac{1-x}{x} + \tilde{c}_2 \frac{e^{-x}}{x}$$

Solution of Exceptional Case 1 ($s_1 = s_2 = c$ and $f(c) = 0$)

Remember, for the first solution, the coefficients are

$$a_k = -\frac{1}{f(c+k)} \sum_{i=1}^{k} g_i(c+k) a_{k-i} \tag{21}$$

where $k = 1, 2, 3, \cdots$. For the second solution, instead of substituting c into s, we can keep s as a variable, i.e.,

$$a_k = a_k(s) \tag{22}$$

Thus,

$$y(s,x) = x^s \sum_{k=0}^{\infty} a_k(s) x^k \tag{23}$$

Substituting it into equation (1), we obtain

$$\mathcal{L}y(s,x) = a_0 f(s) x^{s-2} = a_0 (s-c)^2 x^{s-2} \tag{24}$$

Notice that for the first solution

$$\mathcal{L}y(c,x) = 0 \tag{25}$$

Thus, it can be said that the first solution is

$$y_1(x) = y(c,x) = x^c \sum_{k=0}^{\infty} a_k(c) x^k \tag{26}$$

Differentiate both sides of equation (24) w.r.t. s, i.e.,

$$\frac{\partial}{\partial s}\left[\mathcal{L}y(s,x)\right] = a_0 \frac{\partial}{\partial s}\left[(s-c)^2 x^{s-2}\right] \tag{27}$$

or

$$\mathcal{L}\left[\frac{\partial y(s,x)}{\partial s}\right] = a_0 \left[2(s-c) + (s-c)^2 \ln x\right] x^{s-2} \tag{28}$$

Notice that

$$\mathcal{L}\left[\frac{\partial y(c,x)}{\partial s}\right] = 0 \tag{29}$$

Thus,

$$y_2(x) = \left[\frac{\partial y(s,x)}{\partial s}\right]_{s=c} \tag{30}$$

is the other solution.

[Example]

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = 0$$

Let

$$y = \sum_{k=0}^{\infty} a_k x^{k+s}$$

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-1} + \sum_{k=0}^{\infty}(k+s)a_k x^{k+s-1} - \sum_{k=0}^{\infty} a_k(k+s)x^{k+s} - \sum_{k=0}^{\infty} a_k x^{k+s} = 0$$

$$\sum_{k=0}^{\infty}(k+s)^2 a_k x^{k+s-1} = \sum_{k=0}^{\infty} a_k(k+s+1)x^{k+s} = \sum_{\ell=1}^{\infty} a_{\ell-1}(\ell+s)x^{\ell+s-1}$$

$$s^2 a_0 x^{s-1} + \sum_{k=1}^{\infty}\left[(k+s)^2 a_k - (k+s)a_{k-1}\right]x^{k+s-1} = 0$$

Indicial equation:

$$s^2 = 0$$

Recurrence equation:

$$(k+s)\left[(k+s)a_k - a_{k-1}\right] = 0$$

Since $k+s \neq 0$ for $k = 1, 2, 3, \cdots$ and $s = 0$,

$$a_k = \frac{a_{k-1}}{k+s}$$

$$a_k = \frac{a_{k-1}}{k+s} = \frac{1}{k+s}\frac{a_{k-2}}{k+s-1} = \cdots = \frac{a_0}{(s+k)(s+k-1)\cdots(s+1)} = \frac{s!a_0}{(k+s)!}$$

Therefore,

$$y(x,s) = \sum_{k=0}^{\infty}\frac{s!}{(k+s)!}a_0 x^{s+k}$$

$$y_1(x) = y(x,0) = a_0\sum_{k=0}^{\infty}\frac{x^k}{k!} = a_0 e^x$$

$$y_2(x) = \left[\frac{\partial y(x,s)}{\partial s}\right]_{s=0}$$

Notice that

$$\frac{\partial y(x,s)}{\partial s} = a_0 \sum_{k=0}^{\infty} \frac{\partial}{\partial s}\left[\left(\frac{1}{k+s}\right)\left(\frac{1}{k+s-1}\right)\left(\frac{1}{k+s-2}\right)\cdots\left(\frac{1}{s+1}\right)x^{s+k}\right]$$

$$= a_0 \sum_{k=1}^{\infty} x^{s+k}\frac{\partial}{\partial s}\left[\left(\frac{1}{k+s}\right)\left(\frac{1}{k+s-1}\right)\left(\frac{1}{k+s-2}\right)\cdots\left(\frac{1}{s+1}\right)\right] + a_0 \sum_{k=0}^{\infty} \frac{s!}{(k+s)!}\frac{\partial}{\partial s}x^{s+k}$$

$$= a_0 \sum_{k=1}^{\infty}\left\{-\frac{s!}{(k+s)!}\left[\frac{1}{s+k}+\cdots+\frac{1}{s+1}\right]x^{s+k}\right\} + a_0 \ln x \sum_{k=0}^{\infty}\frac{s!}{(k+s)!}x^{s+k}$$

Thus,

$$y_2(x) = -a_o \sum_{i=1}^{\infty}\frac{x^i}{i!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}\right) + a_0 e^x \ln x$$

$$= a_0\left[e^x \ln x - \sum_{i=0}^{\infty}\frac{x^i}{i!}\varphi(i)\right]$$

$$= a_0\left[y_1(x)\ln x - \sum_{i=0}^{\infty}\frac{x^i}{i!}\varphi(i)\right]$$

where

$$\varphi(i) = \sum_{m=1}^{i}\frac{1}{m}$$

for $i = 1, 2, 3, \cdots$, but $\varphi(i) = 0$ for $i = 0$.

Solution of Exceptional Case 2 ($s_1 - s_2 = \ell$ and ℓ is a positive integer)

First, note that $f(s_1) = f(s_2 + \ell) = 0$. From equation (14)

$$a_k(s) = -\frac{1}{f(s+k)} \sum_{i=1}^{k} g_i(s+k) a_{k-i} \tag{31}$$

Thus,

$$y_1(x) = y(s_1, x) = x^{s_1} \sum_{k=0}^{\infty} a_k(s_1) x^k \tag{32}$$

is a solution.

$$\mathcal{L}y(s, x) = a_0(s - s_1)(s - s_2)x^{s-2} \tag{33}$$

Thus, $\mathcal{L}y(s_1, x) = 0$. Notice also that

$$(s - s_2)\mathcal{L}\left[y(s, x)\right] = \mathcal{L}\left[(s - s_2)y(s, x)\right] = a_0(s - s_1)(s - s_2)^2 x^{s-2} \tag{34}$$

Since

$$\mathcal{L}\left[\frac{\partial}{\partial s}(s - s_2)y(s, x)\right]_{s=s_2} = 0 \tag{35}$$

The second solution can be found by

$$y_2(x) = \left[\frac{\partial}{\partial s}(s - s_2)y(s, x)\right]_{s=s_2} \tag{36}$$

[**Example**]

$$\mathcal{L}y \equiv x\frac{d^2y}{dx^2} - y = 0$$

Let $y = \sum_{k=0}^{\infty} a_k x^{k+s}$

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-1} - \sum_{k=0}^{\infty} a_k x^{k+s} = 0$$

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)a_k x^{k+s-1} - \sum_{\ell=1}^{\infty} a_{\ell-1} x^{\ell+s-1} = 0$$

$$s(s-1)a_0 x^{s-1} + \sum_{k=1}^{\infty}\left[(k+s)(k+s-1)a_k - a_{k-1}\right] x^{k+s-1} = 0$$

Indicial Equation:

$$s(s-1) = 0 \qquad \Rightarrow \qquad s_1 = 1 \qquad s_2 = 0$$

$$s_1 - s_2 = 1$$

Recurrence Equation:

$$a_k = \frac{a_{k-1}}{(k+s)(k+s-1)} = \frac{a_0}{(s+k)[(s+k-1)\cdots(s+1)]^2 s}$$

for $k = 1, 2, 3, \cdots$. Since, when $s = s_2 = 0$, $a_1 \to \infty$, no solution of the assumed form can be found for s_2.

The first solution is obtained by substituting $s = s_1 = 1$, i.e.,

$$y_1(x) = a_0 \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!k!} = a_0 u_1(x)$$

The second solution can be obtained by using the proposed method, i.e.,

$$y_2(x) = \frac{\partial}{\partial s} s y(s,x)|_{s=0}$$

$$= a_0 \left[u_1(x) \ln x + 1 - \sum_{k=1}^{\infty} \frac{\varphi(k) + \varphi(k-1)}{k!(k-1)!} x^k \right]$$

where

$$\varphi(k) = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{m=1}^{k} \frac{1}{m} & \text{if } k = 1, 2, 3, \cdots \end{cases}$$

The derivation of the second solution is given in the sequel:

$$y_2(x) = \left[\frac{\partial}{\partial s} \sum_{k=0}^{\infty} s a_k(s) x^{k+s} \right]_{s=0}$$

$$= \sum_{k=0}^{\infty} \left\{ \frac{d}{ds} \left[s a_k(s) \right] \right\}_{s=0} x^k + \ln x \left[\sum_{k=0}^{\infty} s a_k(s) x^{s+k} \right]_{s=0}$$

The series in the second term on the RHS of the above equation is

$$\left[\sum_{k=0}^{\infty} s a_k(s) x^{s+k} \right]_{s=0}$$

$$= \left\{ s \left[a_0 x^s + \frac{a_0}{(s+1)s} x^{s+1} + \frac{a_0}{(s+2)(s+1)^2 s} x^{s+2} + \frac{a_0}{(s+3)(s+2)^2(s+1)^2 s} x^{s+3} + \cdots \right] \right\}_{s=0}$$

Notice that the first term disappears after multiplying $s(=0)$. However, the s in the denominators of the other terms can be cancelled out. Thus,

$$\left[\sum_{k=0}^{\infty} s a_k(s) x^{s+k} \right]_{s=0} = a_0 \sum_{k=1}^{\infty} \frac{x^k}{k \cdot (k-1)^2 \cdots 2^2 \cdot 1^2} = a_0 \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)! k!} = y_1(x) = a_0 u_1(x)$$

On the other hand, let us consider

$$\Theta_k = \left\{ \frac{d}{ds} \left[s a_k(s) \right] \right\}_{s=0}$$

When $k = 0$,

$$\Theta_0 = \frac{d}{ds} (s a_0) = a_0$$

When $k = 1$,

$$\Theta_1 = \left[\frac{d}{ds} \left(s \frac{a_0}{s(s+1)} \right) \right]_{s=0} = \left[- \frac{a_0}{(s+1)^2} \right]_{s=0} = -a_0$$

When $k = 2, 3, 4, \cdots$,

$$\Theta_k = \left[\frac{d}{ds} \frac{a_0}{(s+k)(s+k-1)^2 \cdots (s+1)^2} \right]_{s=0}$$

Let

$$h(s) = \frac{a_0}{(s+k)(s+k-1)^2 \cdots (s+1)^2}$$

Thus,

$$\Theta_k = \left[-\frac{1}{s+k} h(s) - \frac{2}{s+k-1} h(s) - \cdots - \frac{2}{s+1} h(s) \right]_{s=0}$$

$$= -\left[\left(\frac{1}{s+k} + \frac{1}{s+k-1} + \cdots + \frac{1}{s+1}\right) + \left(\frac{1}{s+k-1} + \cdots + \frac{1}{s+1}\right)\right]_{s=0} h(0)$$

$$= -\left[\varphi(k) + \varphi(k-1)\right] \frac{a_0}{k \cdot (k-1)^2 \cdots 2^2 \cdot 1^2}$$

$$= -\left[\varphi(k) + \varphi(k-1)\right] \frac{a_0}{k!(k-1)!}$$

Therefore, by substituting Θ_0, Θ_1, Θ_2, $\cdots$ back, the second solution becomes

$$y_2(x) = a_0 \ln x u_1(x) + a_0 - a_0 x - \sum_{k=2}^{\infty} a_0 \frac{\varphi(k) + \varphi(k-1)}{k!(k-1)!} x^k$$

$$= a_0 \left[u_1(x) \ln x + 1 - \sum_{k=1}^{\infty} \frac{\varphi(k) + \varphi(k-1)}{k!(k-1)!} x^k\right]$$